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ARCS ON THE CIRCLE AND p-TUPLETS ON THE LINE

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The object of this paper is to study some intersectional properties of certain families of sets on the circle and on the line. On the circle the families consist of arcs, and on the line they consist of unions of p intervals for some $p > 1$. The results are related to the classical theorem of Helly and to a result of Grünbaum and Motzkin.

1. Introduction

A family $\mathcal{F}$ of n sets is said to be $\mathscr{C}(r)$, $1 < r < n$, if the intersection of *every* r members of $\mathcal{F}$ is non-empty. $\mathcal{F}$ is said to be $\mathscr{S}(m)$, $1 < m < n$, if the intersection of *some* m members of $\mathcal{F}$ is non-empty. Finally, $\mathcal{F}$ is said to be $\mathscr{A}(n)$ if the intersection of *all* n members of $\mathcal{F}$ is non-empty.

The linear case of a well-known result of Helly [5] states that if $\mathcal{F}$ is a family of n closed intervals on the line, then $\mathscr{C}(2)$ implies $\mathscr{A}(n)$. We include a very short proof. Let $\mathcal{F} = \{[a_i, b_i] \mid 1 \leq i \leq n\}$. Then $a = \max\{a_i \mid 1 \leq i \leq n\}$ belongs to $\bigcap \mathcal{F}$.

This problem has been modified in at least two ways. First, the line is replaced by the circle, with the intervals now becoming arcs. Secondly, the intervals on the line are replaced by disjoint unions of $2, 3, \ldots, p, \ldots$ intervals, which we shall call twins, triplets, ..., p-tuplets, ..., and so on.

It is of interest to note that, in both modifications, simple counterexamples show that not even $\mathscr{C}(n-1)$ implies $\mathscr{A}(n)$. Two questions are then raised.

(A) What is the maximal number m such that $\mathscr{C}(r)$ implies $\mathscr{S}(m)$?

(B) What further conditions are required to ensure that $\mathscr{C}(r)$ implies $\mathscr{A}(n)$ for some r?

Most results in the literature are of type (B). Hadwiger, Debrunner and Klee [4] have compiled several results on arcs on the circle.

(1) If a family of n arcs is such that each arc is smaller than a semicircle, then $\mathscr{C}(3)$ implies $\mathscr{A}(n)$.

(2) If a family of n arcs is such that each arc is smaller than one-third of a circle, then $\mathscr{C}(2)$ implies $\mathscr{A}(n)$.

Results on twins and triplets on the line are contained in papers by Grünbaum and Motzkin [3] and Larman [8], which deal with the general problem in any finite dimension.

(3) If a family of n twins is such that the intersection of every pair of twins is also a twin, then $\mathscr{C}(4)$ implies $\mathscr{A}(n)$.

(4) If a family of n triplets is such that the intersection of every pair or trio of triplets is also a triplet, then $\mathscr{C}(6)$ implies $\mathscr{A}(n)$.

On the other hand, we found only one result of type (A), mentioned by Hadwiger, Debrunner and Klee [3]: If $\mathscr{F}$ is a family of n arcs on the circle which is $\mathscr{C}(2)$, then there exist a pair of antipodal points such that each arc in $\mathscr{F}$ contains either of them. It follows then that $\mathscr{C}(2)$ implies $\mathscr{S}(\frac{1}{2}n)$.

We pursue the research in this direction. Let $m = m(r, n)$ be the maximal number such that $\mathscr{C}(r)$ implies $\mathscr{S}(m)$ for any family of n arcs on the circle. Thus the above result will read $m(2, n) \geqslant n/2$. We are able to determine $m(r, n)$ explicitly.

$$m(r, n) = n + 1 - \left\{\frac{n}{r}\right\}.$$

The proof is contained in Section 2.

Now let $p > 1$ and let $m = m(p, r, n)$ be the maximal number such that $\mathscr{C}(r)$ implies $\mathscr{S}(m)$ for any family of n p-tuplets on the line. We are only able to determine three non-trivial values.

$$m(2, 2, 5) = m(2, 2, 6) = m(2, 2, 7) = 3.$$

However, we manage to establish the existence of

$$c(p, r) = \lim_{n \to \infty} \frac{m(p, r, n)}{n}$$

and prove that

$$c(p, r) \geqslant (pr)^{1/(1-r)}.$$

For $p = 2$, we have

$$\frac{r-10}{r-1} \leqslant c(2, r) \leqslant \frac{r-1}{r},$$

and for $r = 2$,

$$c(p, 2) \leqslant \frac{1}{p}.$$

Furthermore,

$$\tfrac{1}{3} \leqslant c(2, 2) \leqslant \tfrac{3}{8}$$

and

$$c(2, 3) \geqslant \tfrac{1}{2}.$$

The proofs are given in Section 3.

We point out that most of our results in Section 3 carry over directly to p-tuplets on the circle. In an earlier paper ([6]), we considered a special type of twins on the circle and used the result to solve a problem on common transversals. For a generalization to p-tuplets in higher dimensional Euclidean spaces, see [7].

For other related problems, consult the comprehensive paper of Danzer, Grünbaum and Klee [1] and the excellent bibliography section in Hadwiger, Debrunner and Klee [3].

2. Arcs on the circle

It may be assumed that all arcs are less than the circle. We use the notation $[a, b]$ to denote an arc on the circle where a is the starting point and b the endpoint of the arc when tracing it in the clockwise direction.

Theorem 1. $m(r, rk+1) \leq (r-1)k+1$.

Proof. Let $N = \{a_1, a_2, \ldots, a_{rk+1}\}$ be $rk+1$ distinct points on the circle in clockwise order. Define

$$\mathcal{F} = \{[a_i, a_{(r-1)k+i}] \mid 1 \leq i \leq rk+1\}$$

with $(r-1)k+i$ reduced (mod $rk+1$) if necessary. Clearly $\mathcal{F}$ is not $\mathcal{S}((r-1)k+2)$. Now every arc contains $(r-1)k+1$ of the points in N. Let $F_1, F_2, \ldots, F_r$ be arcs in $\mathcal{F}$. Then there are $r(rk-k+1)$ ordered pairs (a, F_j) where $a \in N \cap F_j$ and $1 \leq j \leq r$. Since

$$(rk+1)(r-1) > r(rk-k+1),$$

it follows that there exists a point $a \in N$ such that $a \in F_j$ for $1 \leq j \leq r$. Hence $\mathcal{F}$ is $\mathscr{C}(r)$ and the proof is completed. □

Theorem 2. $m(r, rk) \geq (r-1)k+1$.

Proof. Suppose $\mathcal{F} = \{F_i \mid 1 \leq i \leq rk\}$ is a family of rk arcs which is $\mathscr{C}(r)$. We shall show that $\mathcal{F}$ is $\mathcal{S}((r-1)k+1)$. Let $F_i = [a_i, b_i]$ with $a_1, a_2, \ldots, a_{rk}$ in clockwise order. By relabelling if necessary, let F_1 be the arc which contains the minimal number of the points $a_1, a_2, \ldots, a_{rk}$; say F_1 contains $a_1, a_2, \ldots, a_t$ for some t, $1 \leq t \leq rk$. If $t \geq (r-1)k+1$, then $a_t \in \bigcap \{F_i \mid 1 \leq i \leq (r-1)k+1\}$ and $\mathcal{F}$ is $\mathcal{S}((r-1)k+1)$. Hence we may assume that $t \leq (r-1)k$.

Let $h_0 = 1$. For $0 \leq j \leq r-3$, h_j having been chosen, define $h_{j+1} = \max\{i \mid a_{h_i} \notin F_i\}$. We have $h_{j+1} \leq t$ as otherwise $F_1 \cap (\bigcap_{i=1}^{j+1} F_{h_i}) = \emptyset$. If $h_{j+1} \leq k + h_j - 1$, then

$$a_{h_j} \in \left(\bigcap_{i=1}^{h_j} F_i\right) \cap \left(\bigcap_{i=k+h_j}^{rk} F_i\right)$$

and $\mathcal{F}$ is $\mathcal{S}((r-1)k+1)$. Hence we may assume that $h_{j+1} \geq k + h_j$.

Consider the point $x = a_{h_{r-2}}$. Now $x \in F_i$ for $i \leq h_{r-2}$ as $h_{r-2} \leq t$. Also $x \in F_i$ for $i > t$ as otherwise $F_i \cap F_1 \cap (\bigcap_{j=1}^{r-2} F_{h_j}) = \emptyset$. Hence x belongs to a total of

$$\begin{aligned} h_{r-2} + (rk - t) &= \sum_{j=1}^{r-3} (h_{j+1} - h_j) + h_1 + (rk - t) \\ &\geq k(r-3) + (k+1) + (rk - (r-1)k) \\ &= (r-1)k + 1 \end{aligned}$$

arcs. This completes the proof of the Theorem. □

Theorem 3.

$$m(r, rk) = (r-1)k + 1$$

and

$$m(r, rk+i) = (r-1)k + i \quad \textit{for} \quad 1 \leq i \leq r-1.$$

Proof. We have $m(r, rk) = m(r, rk+1) = (r-1)k + 1$ from Theorems 1 and 2. The second statement follows from the obvious inequality $m(r, n+1) \leq m(r, n) + 1$ and the pigeonhole principle □

Corollary. $m(r, n) = n + 1 - \{n/r\}$.

3. *p*-tuplets on the line

3.1. *Preliminary results*

We first determine some non-trivial values of $m(p, r, n)$.

Theorem 4. $m(2, 2, 5) = m(2, 2, 6) = m(2, 2, 7) = 3$.

Proof. We first show that $m(2, 2, 7) \leq 3$. Let $\mathcal{F} = \{F_i \mid 1 \leq i \leq 7\}$ be defined by

$$\begin{aligned} F_1 &= [0, 0.5] \cup [3, 5.5], \\ F_2 &= [0, 1.5] \cup [7, 9.5], \\ F_3 &= [0, 2.5] \cup [6, 8.5], \\ F_4 &= [1, 3.5] \cup [11, 11.5], \\ F_5 &= [2, 4.5] \cup [8, 10.5], \\ F_6 &= [4, 6.5] \cup [9, 11.5], \\ F_7 &= [5, 7.5] \cup [10, 11.5]. \end{aligned}$$

It is easy to verify that $\mathcal{F}$ is $\mathscr{C}(2)$ but not $\mathcal{S}(4)$.

To show that $m(2,2,5)\geq 3$, suppose $m(2,2,5)\leq 2$ and let $\mathcal{G}$ be the interval graph on 10 vertices which a suitable choice of 5 twins provides. $\mathcal{G}$ is a forest since it is an interval graph and has no triangles. Thus $\mathcal{G}$ has at most 9 edges. However, each of the $\binom{5}{2}=10$ pairs of twins has non-empty intersection, which requires $\mathcal{G}$ to have at least 10 edges. Thus we have a contradiction. Since

$$3\leq m(2,2,5)\leq m(2,2,6)\leq m(2,2,7)\leq 3,$$

the theorem is proved. □

To establish the existence of $c(p,r)$, first observe that $m(p,r,n+1)\leq m(p,r,n)+1$ and that $m(p,r,kn)\leq km(p,r,n)$. Now we prove a more general result which is adapted from Fekete [2].

Theorem 5. *If $f(n)$ is a function on integral values satisfying $f(n+1)\leq f(n)+1$ and $f(kn)\leq kf(n)$, then $\lim_{n\to\infty} f(n)/n$ exists.*

Proof. Let $\alpha=\lim_{n\to\infty} f(n)/n$. Let $\varepsilon>0$ be given. We shall show that for all sufficiently large n, $f(n)/n<\alpha+\varepsilon$.

Since α is the inferior limit, there exists a t such that $f(t)/t<\alpha+\frac{1}{2}\varepsilon$. For sufficiently large n, write $n=kt+i$, $0\leq i\leq t-1$. Now

$$\begin{aligned} f(n)&=f(kt+i)\leq kf(t)+i\\ &<kt(\alpha+\tfrac{1}{2}\varepsilon)+i=n(\alpha+\tfrac{1}{2}\varepsilon)-i(\alpha+\tfrac{1}{2}\varepsilon-1). \end{aligned}$$

Thus

$$\frac{f(n)}{n}<\alpha+\tfrac{1}{2}\varepsilon-\frac{i(\alpha+\frac{1}{2}\varepsilon-1)}{n}<\alpha+\varepsilon,$$

and the theorem follows. □

Fekete's result actually states that $\lim_{n\to\infty} f(n)/n$ exists if $f(n_1+n_2)\leq f(n_1)+f(n_2)$ for any n_1, n_2. We believe, but cannot prove, that $m(p,r,n)$ is subadditive.

3.2. *Upper bounds for $c(p,r)$*

The proof in Theorem 4 that $m(2,2,7)\leq 3$ is based on a combinatorial idea which we formulate as follows. Consider a sequence Q of length np containing p appearances of each of the elements in $\{1,2,\ldots,n\}$ in an arbitrary order. For any subset A of $\{1,2,\ldots,n\}$, define $d_Q(A)$ to be the minimal length of a consecutive block in Q which contains all elements in A. For $1<r<n$, define $d_Q(r)=\max\{d_Q(A)\mid |A|=r\}$.

Based on Q, we construct a family of n p-tuplets as follows. For $1\leq i\leq n$ and $1\leq j\leq p$, replace the jth appearance of i in Q by a closed interval F_{ij}. Two intervals intersect if and only if the terms of Q they replace lie within a consecutive block in Q of length $d_Q(r)$. Now let $\mathscr{F} = \{F_i \mid 1 \leqslant i \leqslant n\}$ with

$$F_i = F_{i1} \cup F_{i2} \cup \cdots \cup F_{ip}.$$

It is clear that $\mathscr{F}$ is $\mathscr{C}(r)$ but not $\mathscr{S}(d_Q(r)+1)$.

The sequence underlying the proof of $m(2, 2, 7) \leqslant 3$ is

$$1\,2\,3\,4\,5\,1\,6\,7\,3\,2\,5\,6\,7\,4,$$

which is optimal. The following general result utilizes the sequence

$$1\,2 \cdots n\,1\,2 \cdots n.$$

Theorem 6. $c(2, r) \leqslant (r-1)/r$.

Proof. Let $n = rk$ for some $k > 1$. We shall prove the theorem by constructing a family $\mathscr{F} = \{F_i \mid 1 \leqslant i \leqslant rk\}$ of rk twins which is $\mathscr{C}(r)$ but not $\mathscr{S}((r-1)k+2)$.

Define

$$F_i = [i, (r-1)k+i] \cup [rk+i, (2r-1)k+i] \quad \text{for} \quad 1 \leqslant i \leqslant rk.$$

It is easy to see that for $i < j$, $F_i \cap F_j \neq \emptyset$ if and only if $j - i \leqslant (r-1)k$ or $rk + i - j \leqslant (r-1)k$. It follows that $\mathscr{F}$ is not $\mathscr{S}((r-i)k+2)$.

Let

$$N = \{1, 2, \ldots, rk\} = N_1 \cup N_2 \cup \cdots \cup N_r$$

where for $1 \leqslant t \leqslant r$,

$$N_t = \{(t-1)k + h \mid 1 \leqslant h \leqslant k\}.$$

Let $A \subset N$ with $|A| = r$. To show that $\mathscr{F}$ is $\mathscr{C}(r)$, we need show that $\bigcap_{i \in A} F_i \neq \emptyset$. We consider two cases.

(i) $A \cap N_t = \emptyset$ for some t. In this case, $i \in A$ means $i \geqslant tk+1$ or $i \leqslant (t-1)k$. If $i \geqslant tk+1$, then

$$i < rk + 1 \leqslant (r+t-1)k + 1 \leqslant (r-1)k + i.$$

If $i \leqslant (t-1)k$, then

$$rk + i < (r+t-1)k + 1 \leqslant (2r-1)k + i.$$

Hence $(r+t-1)k + 1 \in F_i$ for all $i \in A$.

(ii) $|A \cap N_t| = 1$ for all t. Let $A = \{a_1, a_2, \ldots, a_r\}$ where $a_t \in N_t$ for $1 \leqslant t \leqslant r$. Write $a_t = (t-1)k + h_t$. Let $h = \min\{h_t \mid 1 \leqslant t \leqslant r\}$ and let $x = (t-1)k + h$. For $a_t \in A$ such that $a_t \leqslant x$,

$$rk + a_t \leqslant (r+t-1)k + h \leqslant (2r-1)k + a_t$$

since $h \leqslant a_t$. For $a_t \in A$ such that $x < a_t$, we must have $a_t \geqslant x + k$. Hence

$$a_t < rk + 1 \leqslant (r+t-1)k + h \leqslant (r-1)k + a_t$$

Thus $(r+t-1)k+h \in F_t$ for all $a_t \in A$. This completes the proof of the theorem. □

For $r=2$, Theorem 6 yields $c(2,2) \leq \frac{1}{2}$, which is weaker than $c(2,2) \leq \frac{3}{7}$, implied by Theorem 4. We shall improve this result by a refinement of the argument in Theorem 4.

Theorem 7. $c(2,2) \leq \frac{3}{8}$.

Proof. We shall make use of the following sequence Q.

$$1\,2\,3\,4\,5\,6\,1\,7\,8\,2\,5\,6\,3\,8\,7\,4,$$

where $d_Q(2)=4$. From it, we may obtain directly a family of 8 twins which is $\mathscr{C}(2)$ but not $\mathscr{S}(5)$.

For any integer $k>1$, replace each term in Q by a block of k distinct numbers. Both appearances of a term in Q are identically replaced, while the replacements for distinct terms are disjoint. This yields a new sequence Q' of $8k$ terms, and we now show that $d_{Q'}(2)=3k+1$.

In Q, were it not for the pairs $\{1,4\}$ and $\{5,7\}$, we could have $d_Q(2)=3$. Note that 4 is flanked on both sides in Q by 1, satisfying $d_Q(\{1,4\})=4$ in both directions. An identical situation exists for 5 and 7.

Let a and b be any numbers in the replacements for 1 and 4 respectively. Let a_1 and a_2 denote the first and second appearances of a respectively and b_1 the first appearance of b. We have $d_{Q'}(\{a_1,b_1\})+d_{Q'}(\{b_1,a_2\})=6k+2$. Hence either $d_{Q'}(\{a_1,b_1\}) \leq 3k+1$ or $d_{Q'}(\{b_1,a_2\}) \leq 3k+1$. Consequently we have $d_{Q'}(\{a,b\}) \leq 3k+1$, and a and b may be chosen so that equality holds. Replacements for the terms 5 and 7 in Q are similarly dealt with. Thus we have $d_{Q'}(2)=3k+1$ as claimed.

Now a family of $8k$ twins which is $\mathscr{C}(2)$ but not $\mathscr{S}(3k+2)$ may be obtained. This completes the proof of the theorem. □

We cannot find an upper bound for $c(p,r)$ which depends on both p and r. However, since $c(p,r) \leq c(2,r)$ for $p>2$, Theorem 6 serves the purpose. We do have an upper bound which depends on p.

Theorem 8. $c(p,2) \leq 1/p$.

Proof. Let $\mathscr{G}$ be a complete graph on $\{1,2,\ldots,2p+1\}$. Now $\mathscr{G}$ has an Euler path starting from and ending in $2p+1$. Obtain a sequence Q from this path by deleting all appearances of $2p+1$. Clearly $d_Q(2)=2$ as $\mathscr{G}$ is a complete graph. Hence we may obtain a family $\mathscr{F}$ of $2p$ p-tuplets which is $\mathscr{C}(2)$ but not $\mathscr{S}(3)$. The theorem now follows by making duplicate copies of $\mathscr{F}$ which overlay one another. □

3.3. *Lower bounds for $c(p, r)$*

The proof of the following general lower bound for $c(p, r)$ depends on a simple observation which we have made in Section 1. If $\bigcap_{i=1}^{n} [a_i, b_i]$ is non-empty, then it must contain one of the a's.

Theorem 9. $c(p, r) \geqslant (pr)^{1/(1+r)}$.

Proof. Let $\mathcal{F} = \{F_i \mid 1 \leqslant i \leqslant n\}$ be a family of n p-tuplets which is $\mathscr{C}(r)$. Let

$$F_i = [a_{i1}, b_{i1}] \cup [a_{i2}, b_{i2}] \cup \cdots \cup [a_{ip}, b_{ip}]$$

with

$$a_{i1} < b_{i1} < a_{i2} < b_{i2} < \cdots < a_{ip} < b_{ip}.$$

Let A be a subset of $\{1, 2, \ldots, n\}$ with $|A| = r$. Since $\mathcal{F}$ is $\mathscr{C}(r)$, $\bigcap_{i \in A} F_i$ is non-empty, and must contain at least one of the points a_{ij} for some $i \in A$ and some j, $1 \leqslant j \leqslant p$. Let one such point be labelled A.

Note that there are altogether $\binom{n}{r}$ labels, each being applied to one of the points in $\{a_{ij} \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant p\}$. Hence at least one point a_{st} must be labelled at least $(1/pn)\binom{n}{r}$ times.

Now a_{st} belongs to F_i for any i which belongs to any label on a_{st}. Each label on a_{st} consists of s and $r-1$ other numbers from $\{1, 2, \ldots, n\}$. Let h be the smallest integer such that

$$\binom{h}{r-1} \geqslant \frac{1}{pn}\binom{n}{r}.$$

It follows that a_{st} belongs to at least h p-tuplets. A crude estimate yields

$$h \geqslant (n - r + 1)(pr)^{1/(1-r)},$$

proving the theorem. $\square$

Note that if p is fixed, Theorem 9 shows that $c(p, r)$ tends to 1 as r tends to infinity.

We now turn our attention to twins. First we prove a lower bound for $c(2, r)$ which, for sufficiently large r, is superior to $c(2, r) \geqslant (2r)^{1/(1-r)}$ given by Theorem 9.

Theorem 10. $c(2, r) \geqslant (r - 10)/(r - 1)$.

Proof. Let $\mathcal{F} = \{F_i \mid 1 \leqslant i \leqslant n\}$ be a family of n twins which is $\mathscr{C}(r)$. We may assume that $\mathcal{F}$ is not $\mathscr{C}(r+1)$. By relabelling if necessary, let $F_1 \cap F_2 \cap \cdots \cap F_{r+1} = \emptyset$.

Define $B_j = \bigcap \{F_i \mid 1 \le i \le r+1, i \ne j\}$ for $1 \le j \le r+1$. Note that the B's are disjoint. Let

$$a_1 = \min\{x \mid x \in B_j \quad \text{for some } j,\ 1 \le j \le r+1\}$$

and

$$a_{r+1} = \max\{x \mid x \in B_j \quad \text{for some } j,\ 1 \le j \le r+1\}.$$

By relabelling if necessary, assume that $a_1 \in B_1$ and $a_{r+1} \in B_{r+1}$.

For $2 \le j \le r$, choose $a_j \in B_j$ and again by relabelling if necessary, assume that $a_2 < a_3 < \cdots < a_r$. Note of course that $a_1 < a_2$ and $a_r < a_{r+1}$.

For $2 \le k \le h \le r$, define $D_{k,h} = \bigcap \{F_i \mid 1 \le i \le r+1, i \ne k, h\}$. We claim that $D_{k,h} \subset [a_1, a_{r+1}]$ with the possible exception of at most two of the D's.

Suppose on the contrary $D_{k_i,h_i} \not\subset [a_1, a_{r+1}]$ for $1 \le i \le 3$. Choose $x_i \in D_{k_i,h_i} - [a_1, a_{r+1}]$. Without loss of generality, we may assume that $x_1 \le x_2 < a_1$.

Since $B_j \subset [a_1, a_{r+1}]$ for $2 \le j \le r$, we have $x_1 \ne x_2$ and $x_2 \notin F_{k_2} \cup F_{h_2}$. Since $(k_1, h_1) \ne (k_2, h_2)$, we may assume that $k_1 \ne k_2 \ne h_1$. Now $x_1 \in F_{k_2}$, $x_2 \notin F_{k_2}$, $a_1 \in F_{k_2}$, $a_{k_2} \notin F_{k_2}$ and $a_{r+1} \in F_{k_2}$, with $x_1 < x_2 < a_1 < a_{k_2} < a_{r+1}$. This contradicts the fact that F_{k_2} is a twin.

Without loss of generality, assume that D_{k_1,h_1} and D_{k_2,h_2} are the only two D's not contained in $[a_1, a_{r+1}]$, and that $k_1 < k_2$. Relabel the twins $F_1, F_2, \ldots, F_{r+1}$ as follows. F_{k_1} becomes F_r, and F_{k_2} becomes F_{r+1}. F_i remains undisturbed for $i < k_1$, becomes F_{i-1} for $k_1 < i < k_2$ and becomes F_{i-2} for $i > k_2$.

Now we have $a_1 < a_2 < \cdots < a_{r-1}$, $a_j \in B_j$ for $1 \le j \le r+1$ and $D_{k,h} \subset [a_1, a_{r-1}]$ for $2 \le k < h \le r-2$. For $2 \le k \le r-3$, define

$$E_k = \bigcap \{F_i \mid 1 \le i \le r+1, i \ne k, k+1\}.$$

We claim that $E_k \subset [a_{k-1}, a_{k+2}]$.

Suppose there exists $x \in E_k$ such that $a_1 \le x < a_{k-1}$ or $a_{k+2} < x \le a_{r-1}$. Now $x \notin F_k \cap F_{k+1}$. Without loss of generality, assume that $a_1 \le x < a_{k-1}$ and $x \notin F_k$. Now $a_1 \in F_k$, $x \notin F_k$, $a_{k-1} \in F_k$, $a_k \notin F_k$ and $a_{k+1} \in F_k$, with $a_1 < x < a_{k-1} < a_k < a_{k+1}$. This is a contradiction as F_k is a twin.

Now let F be any twin other than $F_1, F_2, \ldots, F_{r-1}$. Consider $F \cap E_k$ for $2 \le k \le r-3$. This is an intersection of r twins, and thus is non-empty. Since $E_k \subset [a_{k-1}, a_{k+2}]$, it follows that $F \cap [a_{k-1}, a_{k+2}] \ne \emptyset$ for $2 \le k \le r-3$.

Since F is a twin, it must contain at least $(r-1)-9$ of the points $a_1, a_2, \ldots, a_{r-1}$. There are altogether $n-r-1$ twins besides $F_1, F_2, \ldots, F_{r+1}$. Hence at least one of the points must belong to

$$\frac{(n-r-1)(r-10)}{r-1}$$

of these twins as well as to r of the twins $F_1, F_2, \ldots, F_{r+1}$. Since

$$\frac{(n-r-1)(r-10)}{r-1} + r \ge \frac{n(r-10)}{r-1},$$

it follows that

$$c(2, r) \geq \frac{r-10}{r-1}$$

and the theorem is proved. $\square$

For small values of r, the above lower bound is not very meaningful. For $r = 2$, Theorem 9 yields $c(2, 2) \geq \frac{1}{4}$. The following argument which improves the bound to $c(2, 2) \geq \frac{1}{3}$ is due to J. Komlos.

Let $\mathcal{F}$ be a family of n twins which is $\mathscr{C}(2)$. Take the convex hull of each twin in $\mathcal{F}$. By Helly's Theorem, their intersection contains at least one point x.

Let $\mathcal{G} = \{G \in \mathcal{F} \mid x \notin G\}$. If $|\mathcal{G}| \leq 2n/3$, we have finished. Assume therefore that $\mathcal{G} = \{G_i \mid 1 \leq i \leq k\}$ for some $k > 2n/3$. Let $G_i = A_i \cup B_i$ with A_i to the left and B_i to the right of x. If $A_i \cap A_j = \emptyset$ for some i and j, let $C_{ij} = B_i \cup B_j$. C_{ij} is a closed interval as $\mathcal{F}$ is $\mathscr{C}(2)$.

We claim that if C_{ij} and C_{kh} are defined, then they intersect. Assuming the contrary, we have $B_i \cap B_k = B_i \cap B_h = B_j \cap B_k = B_j \cap B_h = \emptyset$. Since $\mathcal{F}$ is $\mathscr{C}(2)$, we must have $A_i \cap A_k \neq \emptyset$, $A_i \cap A_h \neq \emptyset$, $A_j \cap A_k \neq \emptyset$ and $A_j \cap A_h \neq \emptyset$. This cannot happen without violating $A_i \cap A_j = A_k \cap A_h = \emptyset$.

By Helly's Theorem, there exists a point y in $\bigcap \{B_i \cup B_j \mid A_i \cap A_j = \emptyset\}$. Let

$$\mathcal{A} = \{A_i \mid 1 \leq i \leq k, y \notin B_i\} \quad \text{and} \quad \mathcal{B} = \{B_i \mid 1 \leq i \leq k, y \in B_i\}.$$

If $|\mathcal{B}| \geq k/2 > n/3$, we have finished. Assume therefore that $|\mathcal{A}| \geq k/2 > n/3$. We claim that $\mathcal{A}$ is $\mathscr{C}(2)$.

Let A_i and A_j be two distinct intervals in $\mathcal{A}$. If $A_i \cap A_j = \emptyset$, then $y \in B_i$ or $y \in B_j$, a contradiction. By Helly's Theorem once more, $\bigcap \mathcal{A} \neq \emptyset$ and the argument is completed. $\square$

For $r = 3$, we have $c(2, 3) \geq 1/\sqrt{6}$ from Theorem 9. We improve this by an ad hoc argument.

Theorem 11. $c(2, 3) \geq \frac{1}{2}$.

Proof. Let $\mathcal{F} = \{F_i \mid 1 \leq i \leq n\}$ be a family of n twins which is $\mathscr{C}(3)$. Let $F_i = [a_i, b_i] \cup [c_i, d_i]$ with $a_i < b_i < c_i < d_i$, $1 \leq i \leq n$. Let $a = \max\{a_i \mid 1 \leq i \leq n\}$ and $d = \min\{d_i \mid 1 \leq i \leq n\}$. Now $a = a_k$ for some k and $d = d_h$ for some h. Clearly $a_k \leq d_h$ as otherwise $F_k \cap F_h = \emptyset$.

We claim that each F_i contains either a_k or d_h, a stronger assertion than $c(2, 3) \geq \frac{1}{2}$. Suppose F_i does not contain a_k or d_h. Then we must have $b_i < a_k \leq d_h < c_i$ and $F_i \cap [a_k, d_h] = \emptyset$. However, $[a_k, d_h] \supset F_k \cap F_h$, so that $F_i \cap F_k \cap F_h = \emptyset$. This is a contradiction and the theorem is proved. $\square$

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